SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED

REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

(2)

FILE COPY

AD-A198 295

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
...DULE	3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited
...BER(S) City College, CUNY No. MB84-04	5. MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 88-0847
6a. NAME OF PERFORMING ORGANIZATION City College, CUNY 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State, and ZIP Code) 138 Street and Convent Avenue New York, NY 10031	7b. ADDRESS (City, State, and ZIP Code) BLI 410 Directorate of Mathematical & Information Sciences, Bolling AFB, DC, 20032-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR 8b. OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR 84-0095
8c. ADDRESS (City, State, and ZIP Code) BLI 410 Bolling AFB, DC, 20332-6448	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. 61102F PROJECT NO. 2304 TASK NO. A5 WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
On a Correlation Inequality and its Applications

12. PERSONAL AUTHOR(S)
Mark Brown

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
~~Technical~~ ~~Formal~~	FROM 3-87 TO 3-88	March 29, 1988	17

16. SUPPLEMENTARY NOTATION
This document F (overlined) infinity

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Correlation coefficient; inequalities; NBUE, NWUE, IFRA, DFRA and DMRL distributions; Monte-Carlo simulation; renewal theory; reliability theory; record values; maintenance policies; Poisson processes; exponential approximations.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Consider a continuous distribution on $[0,\infty)$ with cdf F, survival function $\bar{F} = 1-F$ and cumulative hazard function $H = -\text{Ln}\bar{F}$. For F NBUE it is shown that the correlation coefficient between $X \sim F$ and $H(X)$ is bounded below by σ/μ, the coefficient of variation of F, while for F NWUE the correlation coefficient is bounded below by μ/σ. Several applications of this inequality and its generalizations are discussed, including Monte-Carlo simulation of the renewal function, exponential approximation of DMRL distributions, moment inequalities for record values and a variance inequality for random event epochs in a homogeneous Poisson process.

F (overlined) delta approx. = delta delta

DTIC ELECTE AUG 29 1988 S E

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT ☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL Maj Brian W. Woodruff	22b. TELEPHONE (Include Area Code) 202-767-5027 22c. OFFICE SYMBOL NM

DD Form 1473, JUN 86 Previous editions are obsolete. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED

88 8 25 091

AFOSR-TR- 88-0847

On a Correlation Inequality and its Applications

By
Mark Brown
The City College, CUNY

March 1988

Accession For	
NTIS GRA&I	X
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

City College, CUNY Report No. MB84-04
AFOSR Technical Report No. 84-04
AFOSR Grant No. 84-0095

Abstract. Consider a continuous distribution on $[0,\infty)$ with cdf F, survival function $\overline{F} = 1-F$ and cumulative hazard function $H = -\mathrm{Ln}\overline{F}$. For F NBUE it is shown that the correlation coefficient between $X \sim F$ and $H(X)$ is bounded below by σ/μ, the coefficient of variation of F, while for F NWUE the correlation coefficient is bounded below by μ/σ. Several applications of this inequality and its generalizations are discussed, including Monte-Carlo simulation of the renewal function, exponential approximation of DMRL distributions, moment inequalities for record values and a variance inequality for random event epochs in a homogeneous Poisson process.

1. Introduction and Summary.

Consider a continuous distribution on $[0,\infty)$, with cdf F, survival function $\overline{F} = 1-F$ and cumulative hazard function $H = -\mathrm{Ln}\overline{F}$. If $X \sim F$ then $H(X)$ is exponentially distributed with mean 1. The random variable $H(X)$ measures lifetime by total hazard overcome until death, while X measures lifetime in ordinary time units. Since H is an increasing function we know that $H(X)$ and X are positively correlated. The question of how positively correlated arose naturally in Brown, Solomon and Stephens (1981) and Brown (1987) in different contexts. In the former paper the asymptotic relative savings in risk between two Monte-Carlo estimators of the renewal function was given by the square of the correlation coefficient between X and $H(X)$. In Brown (1987), a quantity closely related to the correlation coefficient was needed to bound the distance between a DMRL (decreasing mean residual life) distribution and its stationary renewal distribution.

In this paper we show that for X NBUE (new better than used in expectation):

$$(1.1) \qquad \rho(X,H(X)) \geq \frac{\sigma}{\mu}$$

while for X NWUE (new worse than used in expectation):

$$(1.2) \qquad \rho(X,H(X)) \geq \frac{\mu}{\sigma} .$$

The lower bound on the correlation coefficient also aids in bounding the expected waiting time, $E(S_2-S_1)$, between the first and second record values corresponding to an i.i.d. sequence $\{X_i, i \geq 1\}$ with $X_i \sim F$. Using (1.1) and (1.2) we show that for F NBUE:

$$\frac{\sigma^2}{\mu} \leq E(S_2-S_1) \leq \sigma \tag{1.3}$$

and for F NWUE:

$$\mu \leq E(S_2-S_1) \leq \sigma. \tag{1.4}$$

The quantity $E(S_2-S_1)$ represents the expected time to failure after the first minimal repair, and is of interest in the study of maintenance policies.

In section 3.4 inequalities are derived for the moments of higher record values. For example it is shown that if F is IFRA, and S_r is the r^{th} record value then:

$$\frac{\mu_{k+r-1}}{\mu_{r-1}} \leq ES_r^k \leq \binom{k+r-1}{k}\mu_k \tag{1.5}$$

where $\mu_j = \int x^j dF(x)$.

In section 3.5 we show that if $\{T_i, i \geq 1\}$ are the arrival epochs for a homogeneous Poisson process with parameter λ, and N is a stopping time, then $\mathrm{Var}\, T_N \geq \lambda^{-2}$. We further show that among all distributions with failure rate uniformly bounded above by λ, the exponential distribution with parameter λ has the minimum variance for the k^{th} record value, for $k \geq 1$.

2. A Correlation Inequality.

Consider a non-negative random variable X with continuous cdf F. Denote by X^* a random variable with cdf $G(x) = \mu^{-1} \int_0^x \overline{F}(t)dt$, the stationary renewal distribution corresponding to F. Let T denote a random variable with distribution $dF_T(t) = t\mu^{-1}dF(t)$. T is distributed as the length of the interval covering an arbitrary fixed point in a stationary renewal process with interarrival time distribution F (Feller (1971) p. 371). Since the backward recurrence time for a stationary renewal process is distributed as G, we see that:

$$T \sim X | X \geq X^* \tag{2.1}$$

where X^* is independent of X.

Next, consider the record value process corresponding to F. We take an i.i.d. sequence $\{X_i, i \geq 1\}$ with $X_i \sim F$ and define $S_1 = X_1$, $N_2 = \{\min i: X_i > X_1\}$, $S_2 = X_{N_2}$, $N_k = \min\{i: X_i > X_{N_{k-1}}\}$, $S_k = X_{N_k}$, $k = 3,4,\ldots$. The sequence $\{S_i, i \geq 1\}$ generates a non-homogeneous Poisson process with $EN(t) = -Ln\overline{F}(t) = H(t)$ (Shorrock (1972)). Note that:

$$S_2 \sim X | X \geq X' \tag{2.2}$$

where X' is independent of X with the same distribution.

We now derive a useful result. Two proofs are given, as each is instructive.

Lemma 2.1. If F is NBUE (NWUE) then S_2 is stochastically larger (smaller) than T.

Proof 1. F NBUE is equivalent to $X \overset{st}{\geq} X^*$. Now $Z(x) = X|X \geq x$ is stochastically increasing in x $(\Pr(Z(x) > t) = \overline{F}(t \vee x)/\overline{F}(x)$, where $a \vee b = \max(a,b)$, and is thus increasing in x) therefore $S_2 = Z(X)$ is stochastically larger than $T = Z(X^*)$ (see (2.1) and (2.2)). The NWUE case similarly follows.

Proof 2. Consider the record value process $\{S_i, i \geq 1\}$. If X is NBUE then $E(S_k - S_{k-1}|S_1,\ldots,S_{k-1}) \leq \mu$, thus $\{S_n - n\mu, n \geq 1\}$ is a super-martingale. Consider the stopping time N_t+1, which is one plus the number of record values in $[0,t]$. Then:

$$(2.3) \qquad ES_{N_t+1} \leq \mu E(N_t+1) = \mu(H(t)+1) .$$

Define $\delta(t) = E(X-t|X > t) = \mu\overline{G}(t)/\overline{F}(t)$. Then $ES_{N_t+1} = t+\delta(t)$ so (2.3) reduces to:

$$(2.4) \qquad t+\delta(t) \leq \mu(H(t)+1) .$$

Now:

$$(2.5) \qquad \overline{F}_{S_2}(t) = \Pr(N(t) \leq 1) = (H(t)+1)\overline{F}(t)$$

while:

$$(2.6) \qquad \overline{F}_T(t) = \Pr(T > t) = t\mu^{-1}\overline{F}(t) + \overline{G}(t) = \mu^{-1}\overline{F}(t)[t+\delta(t)] .$$

The result now follows from (2.4), (2.5) and (2.6). The NWUE is handled analogously.

Lemma 2.2. If F is NBUE then $\rho(X,H(X)) \geq \sigma/\mu$. If F is NWUE then $\rho(X,H(X)) \geq \mu/\sigma$.

Proof. Note that $dF_{S_2}(t) = H(t)dF(t)$ while $dF_T(t) = t\mu^{-1}dF(t)$. By Lemma 2.1:

$$ES_2 = E(XH(X)) \geq ET = \mu_2/\mu \ . \tag{2.7}$$

Now subtract μ and divide by σ on both sides of (2.7) and the NBUE result follows.

Next, assume that X is NWUE. It follows from Lemma 2.1 that for any increasing function ℓ (with the expectations existing):

$$E\ell(S_2) = \int \ell(x)H(x)dF(x) \leq \mu^{-1}\int x\ell(x)dF(x) = E\ell(T) \tag{2.8}$$

choose $\ell(x) = H(x)$, then:

$$EH^2(X) = 2 \leq \mu^{-1}E(XH(X)) \ . \tag{2.9}$$

From (2.9) the NWUE result easily follows.

3. Applications.

3.1. Monte-Carlo estimation of the renewal function. Suppose we wish to estimate $M(t)$, the expected number of renewals in $[0,t]$ for a renewal process with interarrival distribution F, by Monte-Carlo simulation. An obvious approach is to simulate $N(t)$, the number of renewals in $[0,t]$, K times $(N_1(t),\ldots,N_K(t))$, and to estimate $M(t)$ by the sample mean. In Brown, Solomon and Stephens (1971) an unbiased estimator $M^*(t)$ was proposed and it was shown that as $t \to \infty$ the asymptotic relative savings in risk between $M^*(t)$ and the estimator based on $N(t)$ was given by $\rho^2(X,H(X))$. Lemma (2.2) gives a lower bound on ρ and thus a lower bound on the asymptotic relative savings in risk.

3.2. Exponential approximation of DMRL distributions. Consider a continuous DMRL (decreasing mean residual life) distribution F on $[0,\infty)$ with stationary renewal distribution G. In Brown (1987) it is shown that:

$$\mathcal{D}^*(F,G) = \sup|F(B)-G(B)| \leq 1-EH(X^*) \tag{3.2.1}$$

where $H = -\mathrm{Ln}\overline{F}$, the cumulative hazard function, and $X^* \sim G$. Now:

$$ES_2 = \int (H(t)+1)\overline{F}(t)dt = \mu[1+EH(X^*)]\,. \tag{3.2.2}$$

But F DMRL implies F NBUE, thus (3.2.2) and Lemma 2.1 give:

$$ES_2 = \mu[1+EH(X^*)] \geq ET = \mu_2/\mu \tag{3.2.3}$$

thus:

$$(3.2.4) \qquad EH(X^*) \geq \sigma^2/\mu^2 \ .$$

From (3.2.1) and (3.2.4) we obtain:

$$(3.2.5) \qquad \mathcal{D}^*(F,G) \leq 1-(\sigma^2/\mu^2) \ .$$

The inequality (3.2.5) thus extends the result of Brown (1987) from F IFR to F DMRL. Moreover it follows from (3.2.5), employing the methodology of Brown (1987) that for F IMRL:

$$(3.2.6) \qquad \sup|\overline{F}(t)-e^{-t/\mu}| \leq 1-(\sigma^2/\mu^2) \ .$$

Thus if F is DMRL with coefficient of variation close to 1, then F is approximately exponential.

3.3. The second record value. Consider S_2-S_1 the interarrival time between the first and second record values in a record value process corresponding to F (equivalently the interarrival time between the first and second events in a non-homogeneous Poisson process with $EN(t) = H(t) = -\mathrm{Ln}\overline{F}(t)$). It follows from Lemma 2.1 that F NBUE implies:

$$(3.3.1) \qquad E(S_2-S_1) \geq ET-\mu = \frac{\mu_2}{\mu} - \mu = \frac{\sigma^2}{\mu}$$

while F NWUE leads to $E(S_2-S_1) \leq (\sigma^2/\mu)$.

The quantity $E(S_2-S_1)$ is the expected residual life for an item which is minimally repaired at its first failure. It is of interest in the evaluation and planning of maintenance policies.

Lemma 3.3.1, below, presents an upper bound of σ for $E(S_2-S_1)$, derived without aging assumptions of F. As is done throughout this paper we assume that F is a continuous distribution on $[0,\infty)$.

<u>Lemma 3.3.1</u>. Let $X \sim F$ and g a function on $[0,\infty)$ with $Eg^2(X) < \infty$. Then:

$$|E(g(S_2)-g(S_1))| \leq \sigma_g$$

where σ_g is the standard deviation of $g(X)$. In particular the choice $g(x) = x$ gives:

$$E(S_2-S_1) \leq \sigma$$

where σ is the standard deviation of X.

<u>Proof</u>. $Eg(S_2) = E(g(X)H(X)) = Eg(X)EH(X) + \sigma_g\sigma_{H(X)}\rho(g(X),H(X)) \leq Eg(X)+\sigma_g$. Thus $E(g(S_2)-g(S_1)) \leq \sigma_g$. Substituting $-g$ for g yields $E(g(S_1)-g(S_2)) \leq \sigma_g$ from which the result follows. $\square$

<u>Corollary 3.3.1</u>. For F NBUE, $\sigma^2/\mu \leq E(S_2-S_1) \leq \sigma$. For F NWUE, $\mu \leq E(S_2-S_1) \leq \sigma$.

<u>Proof</u>. The NBUE case follows from expression (3.3.1) and Lemma 3.3.1. The NWUE case follows from Lemma 3.3.1 and the obvious NWUE inequality $E(S_2-S_1) \geq \mu$. $\square$

A function $g(x)$ on $[0,\infty)$ is defined to be starshaped if $\frac{g(x)}{x}$ is increasing (meaning non-decreasing). If g is non-negative and starshaped then g is increasing.

Consider, now, a function g which is non-negative and starshaped on $[0,\infty)$, with $\mu_g = Eg(X) < \infty$. Define:

$$dF_g(t) = g(t)dF(t)/\mu_G .$$

Then: $dF_g(t)/dF_T(t) = \mu_g^{-1}\mu(g(t)/t)$ which is increasing. Thus F_g is larger than F_T under the partial ordering of monotone likelihood ratio (Lehmann [1959] p. 73) and is thus stochastically larger. It follows that:

$$E[Xg(X)] \geq \mu_g\mu_2/\mu . \tag{3.3.2}$$

Now assume that F is NBUE. By Lemma 2.1 and (3.3.2):

$$Eg(S_2) \geq Eg(T) = \mu^{-1}E(Xg(X)) \geq \mu_g\mu_2/\mu^2 . \tag{3.3.3}$$

Thus for F NBUE and g non-negative and starshaped it follows from Lemma 3.3.1 and (3.3.3) that:

$$\frac{\sigma^2}{\mu^2}\mu_g \leq E(g(S_2)-g(S_1)) \leq \sigma_g . \tag{3.3.4}$$

The choice $g(x) = x$ leads to the NBUE inequality of Corollary 3.3.1.

3.4. <u>Higher record values</u>. Let S_k denote the k^{th} record value in a record value process corresponding to F continuous. Since S_k is the k^{th} event epoch in a non-homogeneous Poisson process with $EN(t) = H(t)$ it follows that:

(3.4.1) $$dF_{S_k}(t) = [(H(t))^{k-1}/(k-1)!]dF(t)$$

and also that:

(3.4.2) $$dF_{S_k}(t) = [H(t)/k-1)]dF_{S_{k-1}}(t),\ k \geq 2\ .$$

Consequently (from 3.4.2):

(3.4.3 $$Eg(S_k) = (k-1)^{-1}E[g(S_{k-1})H(S_{k-1})]\ .$$

Now $H(S_{k-1})$ is gamma distributed with parameters $k-1$ and 1 (the sum of $k-1$ i.i.d. exponentials with parameter 1) thus $ES_{k-1} = \text{Var } S_{k-1} = k-1$.

Using the mean and variance of $H(S_{k-1})$, (3.4.3) and the upper bound for the product moment, $EUV \leq EUEV+\sigma_U\sigma_V$ with $U = S_{k-1}$, $V = H(S_{k-1})$ we obtain:

(3.4.4) $$Eg(S_k) \leq Eg(S_{k-1})+(\sigma(g(S_{k-1}))/\sqrt{k-1})\ .$$

From (3.4.4) we obtain the following generalization of Lemma (3.3.1):

(3.4.5) $$|E[(g(S_k)-g(S_{k-1})]| \leq \sigma(g(S_{k-1}))/\sqrt{k-1}\ .$$

The case $k=2$ corresponds to Lemma 3.3.1. However the more general inequality appears to be computationally useful only when $k=2$. For general k $\sigma(g(S_{k-1}))$ is no easier to compute than $E(g(S_k)-g(S_{k-1}))$.

We have no analogue of Lemma 2.1 for F NBUE or NWUE. However if we strengthen the restriction on F from NBUE (NWUE) to IFRA (DFRA) then we obtain the following:

Lemma 3.4.2. Let F be a continuous IFRA distribution, and T_r be a random variable with distribution $dF_{T_r}(t) = x^{r-1} dF(x)/\mu_{r-1}$, where μ_m is the m^{th} moment of F. Then S_r is stochastically larger than T_r and:

$$\frac{\mu_{k+r-1}}{\mu_{r-1}} \leq ES_r^k \leq \binom{k+r-1}{k}\mu_k .$$

If F is a continuous DFRA distribution with finite $(r-1)^{st}$ moment then S_r is stochastically smaller than T_r. If in addition F has finite $(k+r-1)^{st}$ moment then the above inequality reverses.

For $r=2$ the above inequalities hold under the weaker condition that F is NBUE or NWUE.

Proof. Note that $dF_{S_r}(t)/dF_{T_r}(t) = [H(t)/t]^{r-1}$ which is increasing as F is IFRA. Thus S_r is larger than T_r under the monotone likelihood ratio and is thus stochastically larger. Thus:

$$ES_r^k \geq ET_r^k = \mu_{k+r-1}/\mu_{r-1} . \tag{3.4.6}$$

Next:

$$\frac{H^k}{k!}\, dF \overset{st}{\geq} \frac{x^k}{\mu_k}\, dF . \tag{3.4.7}$$

Multiply both sides of (3.4.7) by $H^{r-1}/(r-1)!$ and integrate obtaining:

$$\binom{k+r-1}{k} \geq \frac{1}{\mu_k} ES_r^k . \tag{3.4.8}$$

Thus $ES_r^k \leq \mu_k\binom{k+r-1}{k}$ and this inequality and (3.4.6) yield the IFRA result. The DFRA case similarly follows. By Lemma 2.1, for F IFRA and $r=2$, $S_2 \overset{st}{\geq} T_2 = T$ which is sufficient by our above derivation for (3.4.6) and (3.4.8) to follow (with $r=2$) . □

Note that the various inequalities derived above for record value processes hold more generally for non-homogeneous Poisson processes.

3.5. <u>A variance inequality</u>. Consider an absolutely continuous distribution F with failure rate function $h(t)$ bounded above by $\lambda(h(t) \leq \lambda$ for all $t \geq 0)$. Let S_1 and S_2 denote the first two record values in a record value process corresponding to F. The failure rate function of S_2-S_1 evaluated at t is a mixture of the values $\{h(s), s \geq t\}$ and is thus bounded above by λ for all t. Consequently:

$$E(S_2-S_1) \geq \lambda^{-1} . \tag{3.5.1}$$

By Lemma 3.3.1:

$$E(S_2-S_1) \leq \sigma \tag{3.5.2}$$

where σ is the standard deviation corresponding to F. From (3.5.1) and (3.5.2) we obtain:

$$\sigma^2 \geq \lambda^{-2} . \tag{3.5.3}$$

Thus among all distributions on $[0,\infty)$ with failure rate bounded above by λ, the exponential distribution with parameter λ has smallest variance.

Next, consider a homogeneous Poisson process on $[0,\infty)$ with intensity λ and event epochs $\{T_i, i \geq 1\}$. Let N be a stopping time and consider the random variable T_N, letting h^* denote its failure rate function. Now:

$$(3.5.4) \qquad h^*(t) = \lambda \Pr(T_N = t \mid T_i = t, \text{ for some } i) \leq \lambda .$$

Thus (3.5.3) and (3.5.4) imply:

$$(3.5.5) \qquad \mathrm{Var}(T_N) \leq \lambda^{-2} .$$

Note that λ^{-2} is the variance of T_1 as well as the variance of $T_{N(t)+1}$, the time of the first event after time t. These event epochs have smallest variance among all random event epochs for the Poisson process.

The inequality (3.5.5) holds for a large variety of random variables arising in secondary processes generated by a Poisson process. These include counter models, queues with Poisson input and uniformizable Markov chains.

Also note that if the failure rate of F is uniformly bounded above by λ, then by Lemma 3.3.1 and the argument used to derive (3.5.1):

$$(3.5.6) \qquad \lambda^{-1} \leq E(S_{k+1} - S_k) \leq k^{-1/2}\sigma(S_k) .$$

Thus $\sigma(S_k) \geq k^{1/2}\lambda^{-1}$, the lower bound being achieved in the exponential case. Thus for $k \geq 1$, the exponential distribution with parameter λ minimizes the variance of the k^{th} record value, among all distributions with failure rates uniformly bounded above by λ. Equivalently, consider a non-homogeneous

Poisson process with intensity function $\lambda(t)$ bounded above by λ. Then $\mathrm{Var}\, S_k \geq k\lambda^{-2}$ where S_k is the k^{th} arrival epoch. Thus among all non-homogeneous Poisson processes with intensity functions uniformly bounded above by λ, the homogeneous Poisson process with intensity λ minimizes the variance of S_k, for all $k \geq 1$.

References.

Brown, M. (1987). "Inequalities for distributions with increasing failure rate." Contributions to the Theory and Application of Statistics, A Volume in Honor of Herbert Solomon, p. 3-17. Academic Press, New York.

Brown, M., Solomon, H. and Stephens, M.A. (1981). "Monte Carlo simulation of the renewal function." J. Appl. Prob., 18, 426-434.

Feller, W. (1971). An Introduction to Probability Theory and its Applications, II, 2nd Edition. John Wiley and Sons, New York.

Lehmann, E.L. (1959). Testing Statistical Hypotheses. John Wiley and Sons, New York.

Shorrock, R.W. (1972). "A limit theorem for inter-record times." J. Appl. Prob., 9, 219-223.